An algorithmic solution to the Blotto game using multi-marginal couplings

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Blotto Game

One of the first games by Borel [1921]. Characterizing/computing optimal strategies **still open**

- 2 generals A and B compete for n **battlefields**
- Battlefield i **worth** $v_{A,i}$ to A (and $v_{B,i}$ to B)
- Total **budget** of A: T_A (resp. T_B for B)
- **Allocation** of A: $(x_{A,1}, \dots, x_{A,n})$ s.t. $\sum_i x_{A,i} = T_A$
- If $x_{A,i} > x_{B,i}$ then A **wins** battlefield i
- **Utility** of A: $\sum_i v_{A,i} 1\{A \text{ wins battlefield} i\}$

Allocation $x_{A,i}, x_{B,i}$ must be **random**
Difficulty: budget constraint holds almost surely

Lotto Solution

Same problem, but budget constraint **in expectation**

Optimal solution [Kovenock & Roberson]
On each battlefield, play a **mixture** of Dirac mass and uniform distribution
Weights and lengths have **explicit** forms.

From Lotto to Blotto

Lotto solution gives **optimal marginals** of $x_{A,i}$.
A coupling of $x_{A,i}$ with same marginals but almost sure constraint $\sum x_{A,i} = T_A$ would be **Blotto optimal**

Joint Mixability

[Wang & Wang, Zimin] proved NSC to the existence of coupling between r.v. Z_i s.t. $\sum Z_i = \text{cst} = \mathbb{E} \sum Z_i$

Bounded r.v. such that $Z_i \in [0, \ell_i]$ with decreasing densities are jointly mixable **if and only if**

$$\max \ell_i \leq \mathbb{E} \sum Z_i \leq \sum \ell_i - \max \ell_i$$

Blotto Solution Characterization

Under mild assumptions: marginals of **Lotto** solutions are **jointly mixable** (they satisfy above NSC)

There exists a Blotto optimal solution with the **same explicit marginals** as the Lotto solution

Efficient computations ?

Reduction to 4 random variables

Naïve ε-discretization: $1/\varepsilon^n$ complexity.

- First **reduction** to only one mixture of a Dirac mass and an uniform (plus $n-1$ uniforms)
- Second **reduction** to only 3 uniforms (instead of $n-1$) by explicite couplings between them

Coupling via (adapted) Sinkhorn

- Sinkhorn: easy coupling of 4 marginals.
- Almost sure constraint breaks standard analysis
- Appropriate discretization + few algebraic tricks:
  Adapted Sinkhorn jointly mixes 4 marginals!

Complexity Analysis

Putting all things together, controlling all errors

Under mild assumptions: computing ε-optimal solutions (or ε-Nash equilibria) has a **complexity**

$$\mathcal{O}\left(n^2 + \frac{\log(1/\varepsilon)}{\varepsilon^4}\right)$$

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