Approximately Strategyproof Tournament Rules with Multiple Prizes

Emily Dale, Jessica Fielding, Hari Ramakrishnan, Sacheth Sathyanarayanan, S. Matthew Weinberg

Princeton University

Setup

- A **tournament** T consists of a set of n teams as well as the results of all $\binom{n}{2}$ matches among all pairs of teams.
- A **tournament ranking rule** r is a function that maps tournaments T to a distribution over rankings σ (where $\sigma(i)$ represents the ranking of team i).
- A **prize vector** is a non-increasing vector $\vec{p} \in \mathbb{R}^n$ such that the team ranked j^{th} receives p_j in prize money. In particular, the vector $\vec{p}$ with $p_j = \frac{n-j}{n-1}$ is called the **Borda prize vector**.

Measures of Fairness

- A team is a *Condorcet winner* of a tournament if it beats every other team. A tournament ranking rule is **Condorcet-Consistent** if it outputs a ranking where a Condorcet winner, if one exists, is always ranked first with probability 1.
- Team i *covers* team j if i beats j, and i beats every team that j beats. A tournament ranking rule r is **Cover-Consistent** if whenever i covers j, r outputs a ranking where i is ahead of j with probability 1.

Manipulability

- Let S be a set of teams. Two tournaments T, T' are $S-$**adjacent** if they are identical except for matches between two teams in S.
- We define $\alpha_k^{\vec{p}}(r)$ to be the maximum prize money under $\vec{p}$ that any set of $\leq k$ teams can gain in r by manipulating the underlying tournament T to an $S-$adjacent T'. For a class of prize vectors $\mathcal{P}$ we define $\alpha_k^{\mathcal{P}}(r)$ to be the maximum prize money that any set of $\leq k$ teams can gain in r under any $\vec{p} \in \mathcal{P}$.
- We define $\alpha_k^{\mathcal{P}}$ to be the best bound on manipulability achievable by a Condorcet-Consistent tournament ranking rule against collusions of k teams that holds for all prize vectors in $\mathcal{P}$.

Related Work / Motivation

Previous authors ([1], [2]) have designed tournament rules with $\alpha_2^{\vec{p}} = 1/3$ for $\vec{p} = \langle 1, 0, \ldots, 0 \rangle$. This has been shown to be the best possible result among all Condorcet-Consistent tournament rules. Some manipulability bounds also exist for $k > 2$. As illustrated above, all prior works only consider the case where $\vec{p} = \langle 1, 0, \ldots, 0 \rangle$. However, several modern tournaments offer rewards for teams beyond the winner. For example, the League of Legends Championship Series directly awards a monetary prize to teams based on their final ranking. Our work extends previous works to consider other different prize vectors and establish non-manipulability bounds.

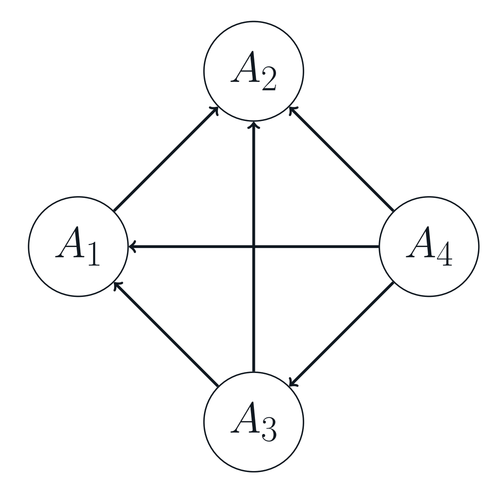
Nested Randomized King of the Hill (NRKotH)

Consider a tournament T on a set S of teams. Let $\sigma_S(u)$ represent the rank of team u. We define the tournament rule NRKotH on this tournament T on $n = |S|$ teams as follows:

1. If $n = 0$, return an empty ordering. Else, continue.
2. Pick a team, u, uniformly at random. Call u the pivot.
3. Let B denote the teams that beat u, and L denote the teams that lose to u.
4. Run NRKotH on B and L, and call the outputs σ_B and σ_L respectively.
5. For all teams $b \in B$, set $\sigma_S(b) := \sigma_B(b)$.
6. Set $\sigma_S(u) := |B| + 1$.
7. For all teams $\ell \in L$, set $\sigma_S(\ell) := \sigma_L(\ell) + |B| + 1$.
8. Output σ_S.

Example

Consider a tournament T on four teams: A_1, A_2, A_3 and A_4. Assume that A_1 defeats A_2 but loses to A_3 and A_4; A_2 loses to A_3 and A_4; and A_3 loses to A_4. This information can be represented as a complete directed graph:



We now consider a simulation of one run of NRKotH as an illustration.

- We first pick a random team as the pivot; say it's A_3.
- Note that $B = \{A_4\}$ is the set of teams that beat A_3 and $L = \{A_1, A_2\}$ is the set of teams that lose to A_3.
- We give A_3 rank $|B| + 1 = 2$.
- We run NRKotH on B. Since $|B| = 1$, we give A_4 rank 1.
- We run NRKotH on L. We first pick a random team as the pivot; say it's A_1.
  - Since A_1 defeats A_2, A_1 gets rank 1 and A_2 gets rank 2 in this sub-tournament.
  - This translates to A_1 getting rank $1 + |B| + 1 = 3$ and A_2 getting rank $2 + |B| + 1 = 4$ in the original tournament.
- The final ranking is thus (A_4, A_3, A_1, A_2).

If the prize vector was the Borda prize vector, the prizes awarded to A_1, A_2, A_3 and A_4 would be $1, 2/3, 0$, and $1/3$ respectively.

Consistence Under Expectation

For any team u and tournament T, we define $w_T(u)$ to be the set of teams that u defeats in tournament T and $\sigma_T^r(u)$ to be the random variable that is the ranking of team u under rule r, applied to T. A tournament rule r is **Consistent under Expectation** if for all n, all tournaments T on n teams, and all u:

$$\sigma_T^r(u) = n - |w_T(u)|$$

Main Result I

For any prize vector in $[0,1]^n$, and any underlying tournament T, under the NRKotH tournament rule, no two teams can manipulate their match to gain expected prize money more than 1/3. Mathematically, we can express this as

$$\alpha_2^{\mathcal{P}}(\text{NRKotH}) = 1/3 = \alpha_2^{\mathcal{P}},$$

where $\mathcal{P}$ denote the set of all prize vectors in $[0,1]^n$. Moreover, this is the best possible guarantee of any Condorcet-Consistent tournament ranking rule.

Main Result II

NRKotH is Consistent under Expectation. As a consequence, for the Borda prize vector $\vec{p}$, no set of k teams can manipulate any of their matches to gain any additional expected prize money. Mathematically, we can express this as

$$\alpha_k^{\vec{p}}(\text{NRKotH}) = 0$$

for all $k \leq n$. Further, for the class of prize vectors $\mathcal{P}^{\varepsilon}$ $\varepsilon-$close to the Borda prize vector (consisting of vectors p where $p_j \in [\frac{n-i}{n-1} - \varepsilon, \frac{n-i}{n-1} + \varepsilon]$), we have that

$$\alpha_k^{\mathcal{P}^{\varepsilon}}(\text{NRKotH}) \leq 2k\varepsilon.$$

Remarks

- Though NRKotH is competitive even with the best Condorcet-Consistent tournament rule, and even with the best guarantee achievable just on $\langle 1, 0, \cdots, 0 \rangle$, it achieves a significantly stronger fairness guarantee: it is Cover-Consistent
- NRKotH is "equivalent" to the quicksort sorting algorithm. Tournament rules equivalent to the mergesort and bubblesort sorting algorithms are not consistent under expectation and do not satisfy Main Result II.

References

[1] Jon Schneider, Ariel Schvartzman, and S. Matthew Weinberg. "Condorcet-Consistent and Approximately Strategyproof Tournament Rules". In: *ITCS 2017*.

[2] Ariel Schvartzman et al. "Approximately Strategyproof Tournament Rules: On Large Manipulating Sets and Cover-Consistence". In: *ITCS 2020*.